Anomalous slow fidelity decay for symmetry breaking perturbations

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Symmetries as well as other special conditions can cause anomalous slowing down of fidelity decay. These situations will be characterized, and a family of random matrix models to emulate them generically presented. An analytic solution based on exponentiated linear response will be given. For one representative case the exact solution is obtained from a supersymmetric calculation. The results agree well with dynamical calculations for a kicked top.

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Sensitivity to perturbations as measured by fidelity decay has received a great deal of attention both in the context of quantum and classical dynamical systems [1, 2, 3, 4] and as a benchmark for the stability of possible quantum information devices [5, 6]. Fidelity may be described as the cross-correlation function between unperturbed and perturbed evolution of a quantum or classical wave system. Linear response theory has been particularly successful in describing fidelity decay, by relating it to the correlation decay of the perturbing operator in the interaction picture [2, 3, 6]. For chaotic systems a recently introduced random matrix model [7] is in excellent agreement with experiments [8, 9].

It has been shown, that so-called residual perturbations that have vanishing diagonal matrix elements in the eigenbasis of the unperturbed Hamiltonian, lead to very slow fidelity decay known also as quantum freeze of fidelity [11, 12]. Fidelity freeze is a pure wave phenomenon without classical analogue [12]. In the field of quantum computation, it is known as *dynamical decoupling* [13, 14]. One can identify four different physical situations in which quantum freeze or anomalous slow fidelity decay occurs. The case when the perturbation can be written as a time derivative (or commutator with the unperturbed Hamiltonian) is treated in Refs. [11, 12].

In this letter we treat the three other and indeed physically most important cases: The first one corresponds to a perturbation which breaks an antiunitary symmetry (e.g. time-reversal) in an optimal way, meaning that the perturbation anticommutes with the antiunitary symmetry, e.g. switching on the magnetic field. The other two cases correspond to a *mean field approach* in which the diagonal part of the perturbation is moved to the unperturbed Hamiltonian. We will consider unperturbed Hamiltonians, with and without an antiunitary symmetry. To obtain a generic understanding, all cases are considered in the framework of random matrix theory (RMT). We present general theoretical results in the linear respons regime, as well as an exact analytical result obtained by a supersymmetry method [18] for the antiunitary symmetry-breaking case. Our results display excellent agreement with numerical experiments for quantum kicked tops.

To study fidelity decay, we consider the perturbed Hamiltonian $H = H_0 + \lambda V$. If $U(t)$ and $U_0(t)$ are the unitary propagators under H and H_0, respectively, we define the fidelity amplitude as

$$f(t) = \langle \Psi_0(t)|\Psi(t)\rangle = \langle \Psi(0)|U_0(-t)\, U(t)|\Psi(0)\rangle \,, \quad (1)$$

where $\Psi(t) = U(t)\,\Psi(0)$ and $\Psi_0(t) = U_0(t)\,\Psi(0)$. Fidelity is defined as $F(t) = |f(t)|^2$. In Refs. [2, 3], it has been shown that within second order time-dependent perturbation theory (Born series) the fidelity amplitude, averaged over random initial states, can be expressed in terms of the two-point time correlation integral $\mathcal{C}(t)$:

$$\langle f(t)\rangle_E = 1 - 4\pi^2\lambda^2\, \mathcal{C}(t) + \mathcal{O}(\lambda^4)\,, \quad (2)$$

where $\langle \ldots \rangle_E$ denotes the average over random initial states, which are concentrated on a small energy interval that contains many levels N_E. For the fidelity amplitude, this averaging amounts to taking a restricted trace of the echo operator, in the eigenbasis of the unperturbed Hamiltonian H_0. Therefore, we may write:

$$\mathcal{C}(t) = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \langle V(t')V(t'')\rangle_E \quad (3)$$

$$\langle V(t')V(t'')\rangle_E = \frac{1}{N_E}{\sum_\alpha}' \sum_\beta |V_{\alpha\beta}|^2\, e^{2\pi \mathrm{i}(E_\alpha - E_\beta)(t'-t'')},$$

where the E_α are the eigenvalues of H_0, and $V(t) = U_0(t)^\dagger\, V\, U_0(t)$ is the perturbation in the interaction picture. The matrix elements $V_{\alpha\beta}$ are taken in the eigenbasis of H_0. The primed sum runs over N_E eigenstates of H_0.

Often, the exponentiated version of Eq. (2),

$$\langle f(t)\rangle_E = e^{-\varepsilon \mathcal{C}(t)} + \mathcal{O}(\lambda^4)\,, \qquad \varepsilon = 4\pi^2\lambda^2\,, \quad (4)$$

is able to describe the fidelity decay from the perturbative to the golden rule regime well [7, 8, 9, 10]. This is true, in particular, if the system is chaotic, such that the perturbation can be described [7] by one of the random Gaussian ensembles (RMT approach) [15].

We now consider the situation of a residual perturbation, i. e., one with vanishing diagonal, $V_{\alpha\alpha} \equiv 0$ within the RMT models. Thus, we assume that the non-zero matrix elements of this perturbation are independent normalized Gaussian random variables with variance

$$\langle |V_{\alpha\beta}|^2 \rangle = 1 - \delta_{\alpha\beta} \ , \tag{5}$$

where $\langle \ldots \rangle$ denotes an ensemble average. For the perturbation matrix V, we consider three different ensembles, which fulfill Eq. (5): (i) an ensemble of imaginary antisymmetric matrices, (ii) an ensemble of real symmetric matrices with deleted diagonal, and (iii) an ensemble of Hermitean matrices with deleted diagonal.

The average of the correlation integral $\mathcal{C}(t)$ over any of those ensembles gives:

$$\langle \mathcal{C}(t) \rangle = \frac{t}{2} - \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' b(t'') \tag{6}$$

where $b(t)$ is the two-point spectral form factor of H_0. Here, as well as throughout the rest of the letter, we use units, where the Heisenberg time is equal to one. If N_E is sufficiently large, $b(t)$ tends to a well defined smooth function (self-averaging); else we average over a random matrix ensemble for H_0 as well. For Gaussian orthogonal (GOE) and unitary (GUE) ensembles, the form factors are given in Ref. [15]. Note that the term proportional to t^2 is missing as compared to the linear response result for a generic perturbation [7]. For what follows, we assume that the average $\langle \ldots \rangle$ includes such a procedure.

Let us first concentrate on the special case of unperturbed Hamiltonians H_0 taken from the GUE. Then, we have $b(t) = \max\{1 - t, 0\}$, and

$$\langle \mathcal{C}(t) \rangle = \mathcal{C}_{\mathrm{GUE}}(t) = \begin{cases} \frac{t}{2} - \frac{t^2}{2} + \frac{t^3}{6} & : t \leq 1, \\ \frac{1}{6} & : t > 1 \end{cases} \tag{7}$$

As a result, for times longer than the Heisenberg time, Eq. (2) predicts the fidelity to "freeze" on a minimal value

$$f_{\mathrm{plateau}} = 1 - \frac{\varepsilon}{6} \ . \tag{8}$$

Since the next correction term grows quadratically in time, $\langle f(t) \rangle = f_{\mathrm{plateau}} + \mathcal{O}(\lambda^4 t^2)$, we find that the plateau ends at a time of order $t^* \sim 1/\lambda$.

Second, we choose H_0 from the GOE. Also in this case, the integrals in equation (6) can be performed analytically (see Ref. [7]). Here we just give the the leading asymptotics for $t \gg 1$:

$$\mathcal{C}_{\mathrm{GOE}}(t) = \frac{\ln(2t) + 2}{12} + \mathcal{O}(t^{-1} \ln t) \tag{9}$$

This yields a logarithmically slow decay of fidelity

$$\langle f(t) \rangle \approx 1 - \frac{\varepsilon}{12} \left[2 + \ln(2t)\right] + \mathcal{O}(\lambda^4 t^2) \ . \tag{10}$$

Both results for the plateau value of the fidelity amplitude, Eq. (8) and Eq. (10), follow from Eq. (6). They are thus valid for any of the three ensembles used for the perturbation, (i), (ii), and (iii). Indeed, a similar result could be obtained for the Gaussian symplectic ensemble.

In [7] the averages $\langle F(t) \rangle$ and $|\langle f(t) \rangle|^2$ were shown to differ only in a term proportional to t^2, which is absent in the quantum freeze case. Thus, the eqs. (2) and (6) yield the linear response approximation for fidelity decay. Numerics indicate that $\langle F(t) \rangle \approx |\langle f(t) \rangle|^2$, in general, in accordance with an argument given in [4].

For long times and small perturbations, the fidelity amplitude can be expressed in terms of level shifts. Within the second order stationary perturbation theory, its average is then given by the Fourier transform of the level curvature distribution [13] which was obtained analytically in [19]. The final result is surprisingly simple:

$$\langle f(t) \rangle = \begin{cases} \tau \ K_1(\tau) & : \mathrm{GOE} \\ (1 + \tau) \ \mathrm{e}^{-\tau} & : \mathrm{GUE} \end{cases} \qquad \tau = \frac{\varepsilon t}{2} \ . \tag{11}$$

where K_1 is the modified Bessel function of first order. The GOE branch is valid for an unperturbed GOE Hamiltonian, and a perturbation matrix of type (i) or (ii). The GUE branch is valid for an unperturbed GUE Hamiltonian, and a perturbation matrix of type (iii). More details on the asymptotic behavior of fidelity decay in situations of quantum freeze will be published elsewhere [13].

One should stress that diagonal elements of the perturbation vanish also in the presence of a discrete or continuous *unitary* symmetry R, of H_0, which anti-commutes with V, $RV = -VR$. However, it turns out that its effect on fidelity enhancement is less drastic than the predictions of Eqs. (7) and (9), because of the lack of correlations between different subspectra of H_0. As a result, the asymptotic growth of the correlation integral is linear $\mathcal{C}(t) \propto t$, for times before and after the Heisenberg time.

For the case of H_0 taken from the GOE and a purely imaginary antisymmetric perturbation the average fidelity can be obtained exactly by supersymmetry techniques in the limit of large dimension N. The calculation is technically much more involved than for the case of a GOE perturbation [16] and will be presented elsewhere [18]. The result again is a VWZ-like integral (see Ref. [17], Eq. (8.10)) and is given by

$$\begin{aligned} \langle f(t) \rangle = 2 \int\limits_{\mathrm{Max}(0,t-1)}^{t} du \int\limits_{0}^{u} & \frac{v \, dv}{\sqrt{[u^2 - v^2][(u+1)^2 - v^2]}} \\ & \times \frac{(t-u)(1-t+u)}{(v^2 - t^2)^2} [1 + \varepsilon(t^2 - v^2)] \\ & \times [t(2u + 1 - t) + v^2] e^{-\frac{\varepsilon}{2}[t(2u+1-t) - v^2]} \ . \end{aligned} \tag{12}$$

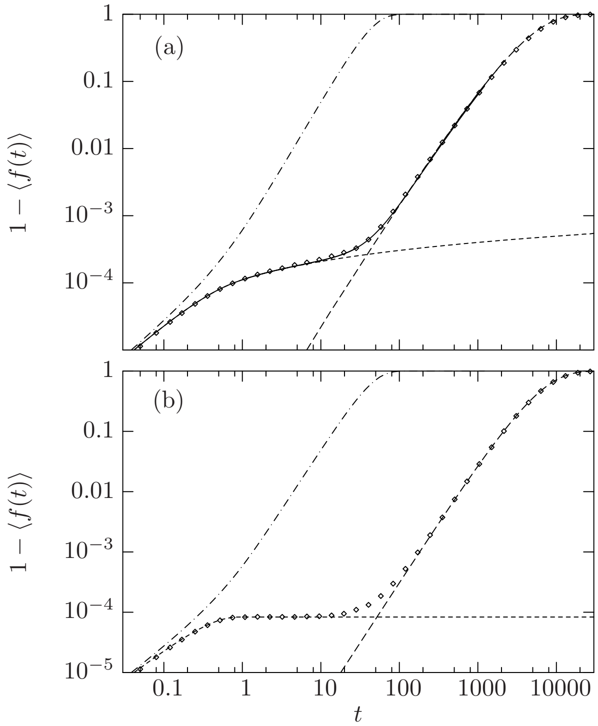

FIG. 1: The complement of the average fidelity amplitude, for a weak perturbation, $\varepsilon = 5\times10^{-5}$. Part (a) shows the GOE case with a purely antisymmetric random Gaussian perturbation. Part (b) shows the GUE case, with an independent GUE perturbation with deleted diagonal. The exponentiated linear response approximations are plotted with short dashed lines (for comparison, the exact results for full GOE and GUE perturbations are shown with chain curves). The results from time-independent perturbation theory are plotted with long dashed lines. In the GOE case, the exact analytical result, Eq. (12) is plotted with a solid line. The random matrix simulations are plotted with points.

The only difference to Ref. [16], where a GOE perturbation was considered, is the additional factor $[1+\varepsilon(t^2-v^2)]$ in the integrand, and a minus sign with the v^2 term in the exponent, where in the GOE case there is a plus sign.

Figure 1 shows the fidelity decay both for H_0 taken from the GOE (a) and the GUE (b), for a small perturbation $\varepsilon = 5\times10^{-5}$. It compares different theoretical approaches. To this end, one minus the average fidelity amplitude is plotted on a double-log scale. The figure shows the exact result calculated from Eq. (12) (in the GOE case only), together with the results from the exponentiated linear response approximation (4) and the asymptotic expression (11). For comparison the fidelity decay for the case of a GOE perturbation [16] is shown as well. We see that the linear response approximation is able to describe the fidelity decay for quite a long time very well. Immediately beyond the time when the linear response formulae fail, the asymptotic results (11) describe fidelity decay quite well. We also compare to numerical random matrix simulations, where we computed averages over 10^4 samples of 100×100 matrices. Only the 10 central states have been taken into account. In the GOE case, we have also performed numerical simulations for symmetric perturbations with deleted diagonal (case (ii)) and we have not found significant deviations from the antisymmetric case (i). For strong perturbations the plateau disappears and we find a partial revival of fidelity near the Heisenberg time, similar as in Ref. [16].

We have concentrated on the unitary antisymmetric perturbation, because of its invariance properties, which allow to obtain results in closed form. Yet the linear response result in Eq. (2) can be carried to higher order, and at least up to sixth order they coincide with the one for symmetric perturbations with deleted diagonal [13].

The RMT model can also be compared to dynamical systems with chaotic classical limit. For this purpose we have considered a quantized kicked top [20].

In the first example, we choose a one step propagator

$$U_\lambda = P^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\gamma S_{\mathrm{y}}}P^{\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\lambda S_{\mathrm{x}}}, \quad \gamma = \pi/2.4 \qquad (13)$$

with $P = \mathrm{e}^{-\mathrm{i}\alpha S_z^2/2S}\mathrm{e}^{-\mathrm{i}S_{\mathrm{z}}}$ and $S_{\mathrm{x,y,z}}$ being standard spin operators. U_0 is time-reversal invariant, and the perturbation S_{y} is antisymmetric in the eigenbasis of U_0. The "symmetrization" of U_0 is essential for V to anticommute with the time-reversal symmetry.

We choose the spin $S = 200$, one initial random state and average the fidelity over 400 realizations of the propagator U_λ where for each realization we draw a parameter α from a Gaussian distribution of width 1 centered around 30. The results of fidelity decay for different strengths of perturbation are shown in Fig. 2(a). We find good agreement with the square of the theoretical result (12) for the fidelity amplitude, which in turn agrees well with RMT simulations for the fidelity (not shown). Without averaging over an ensemble of dynamical systems we get considerable fluctuations around RMT curves. Note that we do not use any fit parameters. The dimensionless perturbation strength ε in Eq. (12) is obtained as $\varepsilon = 2N\sigma_{\mathrm{cl}}(S\lambda)^2 = 4\lambda^2 S^3\sigma_{\mathrm{cl}}$, where $\sigma_{\mathrm{cl}} = 0.153$ is an integral of the classical correlation function calculated using the corresponding classical map, see *e.g.* Ref. [3] for more details. Heisenberg time is $t_{\mathrm{H}} = N = 2S$.

We also consider an unperturbed propagator without time-reversal symmetry, that corresponds to GUE case,

$$U_\lambda = P\mathrm{e}^{-\mathrm{i}\gamma S_{\mathrm{y}}}\,\mathrm{e}^{-\mathrm{i}\mu S_{\mathrm{x}}^2/2S}\,\mathrm{e}^{-\mathrm{i}\xi S_{\mathrm{x}}}\,\mathrm{e}^{-\mathrm{i}\lambda S'_{\mathrm{x}}}. \qquad (14)$$

with $\gamma = \pi/2.4, \mu = 10, \xi = 1$. Here we have set diagonal matrix elements of the perturbation in the eigenbasis of U_0 to zero by hand, $S'_{\mathrm{x}} = S_{\mathrm{x}} - \mathrm{diag}S_{\mathrm{x}}$. We take $S = 200$ and average the fidelity over 100 samples, similarly as for GOE case. As above we determine ε from the classical correlation integral $\sigma_{\mathrm{cl}} = 0.168$. In Fig. 2b the results of the numerical simulation are plotted, together with

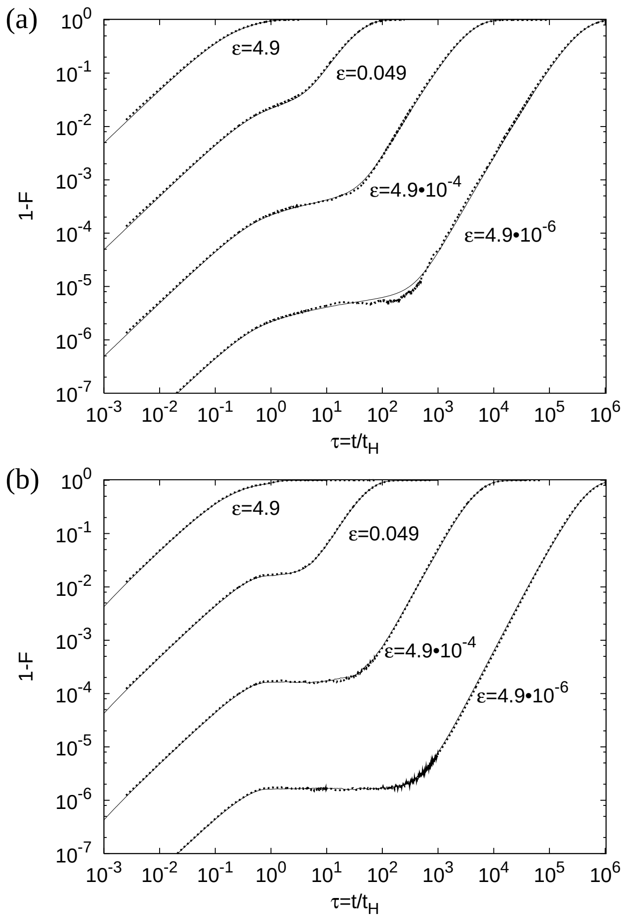

FIG. 2: Fidelity freeze for a quantized kicked top. In (a) perturbation is imaginary antisymmetric with unperturbed dynamics having antiunitary symmetry (13) while in (b) the unperturbed evolution as well as the perturbation have no symmetries left (14). The dashed lines give the numerical simulations, while the solid lines give the square of the theoretical prediction (12) in (a) and RMT simulations in (b).

RMT Monte Carlo simulation (full line). Again, good agreement with the RMT model is observed.

We have presented RMT models that display the eminent features of quantum freeze of fidelity under a wide range of circumstances not previously considered. We allow for any unperturbed Hamiltonian or ensemble of Hamiltonians for which the spectral form factor is known. The perturbations are represented by ensembles of random Hermitean matrices with zero entries on the diagonal. We give a perturbative solution for the general model, and we present an exact solution obtained by supersymmetric techniques, for the case of Hermitean antisymmetric perturbations of GOE Hamiltonians. Kicked top models display excellent agreement with the random matrix results.

The physical importance of such systems becomes apparent in two quite different aspects. On one hand mean field theories in some sense include the diagonal part of the perturbation in the unperturbed Hamiltonian, and thus the quantum freeze sheds new light on the surprising success of such theories. On the other hand this result shows that for a quantum information process to be effective beyond the Heisenberg time, one has to suppress the diagonal part of any static perturbation.

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